# GEOMETRY ON THE PLANE (4)
## QUADRILATERALS INSCRIBED IN A CIRCLE
## &
## CIRCUMSCRIBED QUADRILATERALS
## VOCABULARY, PROPERTIES & EXERCISES

1. QUADRILATERALS INSCRIBED IN A CIRCLE

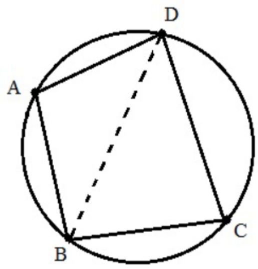
   Regular geometric objects can be always imaged together with their inscribed or circumscribed circles. Another type of objects inscribed in a circle are the so - called **cyclic quadrilaterals**. **This means that the vertices are lying on the same circle**.

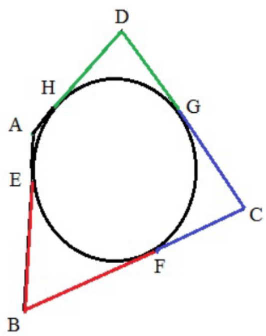
   **Theorem 1** *The quadrilateral $ABCD$ is a cyclic quadrilateral if and only if the sums of the opposite angles are equal.*

2. CIRCUMSCRIBED/TANGENTIAL QUADRILATERALS

   A quadrilateral is called **tangential** if **it has an inscribed circle which touches all the sides of the quadrilateral**.

   **Theorem 2** *A convex quadrilateral is tangential if and only if the sum of the opposite sides are equal.*




3. EXERCISES

   **Exercise 1** *Draw a rhombus around a circle of area* $100$*, so that the rhombus has an angle* $30^\circ$*. Calculate the area of the rhombus.*

   **Exercise 2** ***[USAMO 1992]*** *Let $ABCD$ be a convex quadrilateral such that the diagonals $AC$ and $BD$ are perpendicular, and let $P$ be their intersection. Prove that the reflections of $P$ with respect to $AB$, $BC$, $CD$, $DA$ lie on a circle.*

   **Exercise 3** ***[Bulgaria 1993]*** *A parallelogram $ABCD$ with an acute angle $BAD$ is given. The bisector of $\angle BAD$ intersects $CD$ at point $L$, and the line $BC$ at point $K$. Prove that the circumcenter of $\Delta LCK$ lies on the circumcircle of $\Delta BCD$.*

**Exercise 4** ***(Ptolemy)*** *Let $ABCD$ be a convex cyclic quadrilateral. Proof that*

$$|AB| \cdot |CD| + |BC| \cdot |DA| = |AC| \cdot |BD|.$$

**Exercise 5** ***(Brahmagupta)****. Proof that if a cyclic quadrilateral has sides $a, b, c, d$ and area $K$, then*

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

*where $s = \frac{a+b+c+d}{2}$ is the semiperimeter.*

**Exercise 6** ***(Brahmagupta)*** *Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals. Then the line through the intersection of the diagonals and the midpoint of any side is perpendicular to the opposite side.*

**Exercise 7** *Brahmagupta's formula implies that the area of a cyclic quadrilateral depends only on the lengths of the sides and not the order in which they occur. Can you demonstrate this fact by 'slicing and dicing'?*

**Exercise 8** *Use Ptolemy's theorem and the previous problem to give a formula for the lengths of the diagonals of a cyclic quadrilateral in terms of the lengths of the sides.*

**Exercise 9** *Let $ABCD$ be a cyclic quadrilateral. Prove that the incenters of triangles $ABC$, $BCD$, $CDA$, $DAB$ form a rectangle.*

**Exercise 10** *With the same notation, prove that the sum of the inradii of $ABC$ and $CDA$ equals the sum of the inradii of $BCD$ and $DAB$.*

SOURCES:


1. C. Vincze & L.Kozma, 2014, 'College Geometry' http://math.unideb.hu/.
2. J. Roe, 1993, 'Elementary Geometry', Oxford University Press.
3. V. Prasolov, 'PLANE GEOMETRY PART 1', Translated from the Russian by D. Leites
4. H. S. M. Coxeter, 1969, 'Introduction to Geometry', John Wiley and Sons, Inc., Second Edition, .