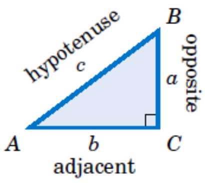
1.9 GEOMETRY ON THE PLANE (3) – TRIGONOMETRY THEORY



1. **Definitions:**
   Consider a right triangle $\triangle ABC$, with the right angle at C and with lengths a,b, and c, as in the figure on the right. For the acute angle A, call the leg BC its **opposite side**, and call the leg AC its **adjacent side**. Recall that the hypotenuse of the triangle is the side AB. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the six **trigonometric functions** of A as follows:

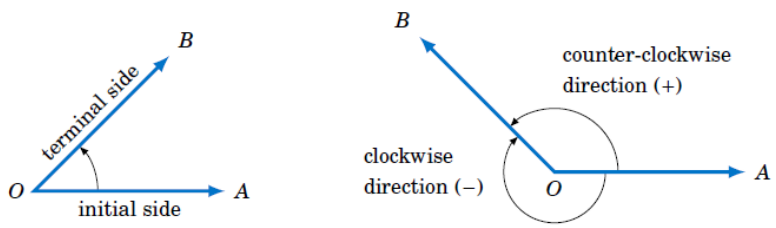
Name of function	**Abbreviation**	**Definition**
sine A	$\sin A$	$= \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c}$
cosine A	$\cos A$	$= \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c}$
tangent A	$\tan A$	$= \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b}$
cosecant A	$\csc A$	$= \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{c}{a}$
secant A	$\sec A$	$= \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{c}{b}$
cotangent A	$\cot A$	$= \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{b}{a}$

2. **Trigonometric functions of** 30° 45° 60° **angle**

	0^o	30^o	45^o	60^o	90^o
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cotg	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

3. **Trigonometric Functions of Any Angle**

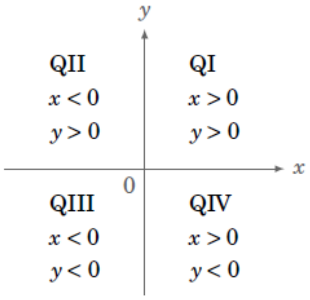
To define the trigonometric functions of any angle - including angles less than 0° or greater than 360° - we need a more general definition of an angle. We say that an angle is formed by rotating a ray $\vec{OA}$ about the endpoint O (called the vertex), so that the ray is in a new position, denoted by the ray $\vec{OB}$. The ray $\vec{OA}$ is called the **initial side** of the angle, and $\vec{OB}$ is **the terminal side** of the angle.



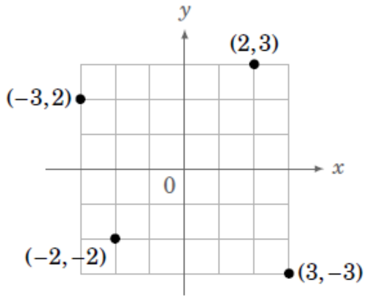
(a) angle $\angle AOB$ (b) positive and negative angles

We denote the angle formed by this rotation as $\angle AOB$, or simply $\angle O$, or even just O. If the rotation is counter-clockwise then we say that the angle is **positive**, and the angle is **negative** if the rotation is clockwise. One full counter-clockwise rotation of $\vec{OA}$ back onto itself (called a **revolution**), so that the terminal side coincides with the initial side, is an angle of 360°; in the clockwise direction this would be $-360°$. Not rotating $\vec{OA}$ constitutes an angle of 0°. More than one full rotation creates an angle greater than 360°.

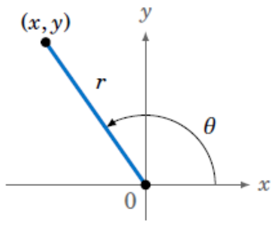
We can now define the trigonometric functions of any angle in terms of **Cartesian coordinates**. Recall that the xy**-coordinate plane** consists of points denoted by pairs (x, y) of real numbers. The first number, x, is the point's x **coordinate**, and the second number, y, is its y **coordinate**. The x and y coordinates are measured by their positions along the x**-axis** and y**-axis**, respectively, which determine the point's position in the plane. This divides the xy-coordinate plane into four quadrants (denoted by QI, QII, QIII, QIV), based on the signs of x and y.



(a) Quadrants I-IV



(b) Points in the plane



(c) Angle θ in the plane

Now let θ be any angle. We say that θ is in standard position if its initial side is the positive x-axis and its vertex is the origin $(0, 0)$. Pick any point (x, y) on the terminal side of ? a distance $r > 0$ from the origin. (Note that $r = \sqrt{x^2 + y^2}$.) We then define the trigonometric functions of θ as follows:

$\sin\theta = \dfrac{y}{r}$	$\cos\theta = \dfrac{x}{r}$	$\tan\theta = \dfrac{y}{x}$
$\csc\theta = \dfrac{r}{y}$	$\sec\theta = \dfrac{r}{x}$	$\cot\theta = \dfrac{x}{y}$

As in the acute case, by the use of similar triangles these definitions are well-defined (i.e. they do not depend on which point (x, y) we choose on the terminal side of θ). Also, notice that $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$, since $|y| \leq r$ and $|x| \leq r$ in the above definitions.