DIFFERENTIAL CALCULUS (2)

DERIVATIVES

THEORY & PROBLEMS

1. VOCABULARY

   (a) **Derivative**. The derivative of a function f at a number x_0 is
   $$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$
   if this limit exists.

   (b) **Tangent Line** An equation of the tangent line to $y = f(x)$ at $(x_0; f(x_0))$ is given by $y - f(x_0) = f'(x_0)(x - x_0)$.

   (c) **Product and Quotient Rules** If f and g are both differentiable, then
   $$(fg)' = f' \cdot g + f \cdot g'$$
   and
   $$\left(\frac{f}{g}\right)' = \frac{f' \cdot g + f \cdot g'}{g^2}$$
   with $g(x) \neq 0$.

   (d) **Absolute Maximum and Minimum** A function f has an absolute maxi- mum at c if $f(c) \geq f(x)$ for all $x \in D$, the domain of f. The number $f(c)$ is called the maximum value of f on D.

   A function f has an absolute minimum at c if $f(c) \leq f(x)$ for all $x \in D$, the domain of f. The number $f(c)$ is called the minimum value of f on D.

   (e) **Local Maximum and Minimum** A function f has a local maximum at c if $f(c) \geq f(x)$ for all x in an open interval containing c.

   A function f has a local minimum at c if $f(c) \leq f(x)$ for all x in an open interval containing c.

   (f) **Extreme Value Theorem** If f is continuous on a closed interval $[a; b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $cd, \in [a; b]$.

   (g) **Fermat's Theorem** If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

   (h) **Critical Number** A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

   (i) **Closed Interval Method** To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a; b]$:

      i. Find the values of f at the critical numbers of f in $(a; b)$.

      ii. Find the values of f at the endpoints of the interval.

      iii. The largest of the values from Step 1 and Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

   (j) **The First Derivative Test** Suppose that c is a critical number of a continuous function f.

i. If f' changes from positive to negative at c, then f has a local maximum at c.
ii. If f' changes from negative to positive at c, then f has a local minimum at c.
iii. If f' does not change sign at c, then f has no local minimum or maximum at c.

2. PROBLEMS

(a) Find the derivatives of the functions.

i. $f(x) = 5x^3 + 12x^2 - 15$
ii. $f(x) = -4x^5 + 3x^2 - \frac{5}{x^2}$
iii. $f(x) = 5(-3x^2 + 5x + 1)$
iv. $f(x) = (x + 1)(x^2 + 2x - 3)$
v. $f(x) = x^3(x^3 + 5x + 10)$
vi. $f(x) = (x^2 + 5x - 3)(x^5 - 6x^3 + 3x^2 - 7x + 1)$
vii. $f(x) = \frac{x^2-x-1}{x+3}$
viii. $f(x) = \frac{x^3+5x^2+2x-1}{x^3+x^2+5x+1}$

(b) Find an equation for the tangent line to $f(x) = \frac{x^2-4}{5-x}$ at $x = 3$.

(c) Find an equation for the tangent line to $f(x) = \frac{x-2}{x^3+4x-1}$ at $x = 1$.

(d) The curve $y = \frac{1}{1+x^2}$ is an example of a class of curves each of which is called a *witch of Agnesi*. Sketch the curve and find the tangent line to the curve at $x = 5$.

(e) Find the derivative of the following function using the definition of the derivative $f(x) = \frac{x}{x+1}$.

(f) Is $f(x) = 2x^3 + \frac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = 2$?

(g) A function is given by $f(x) = 3x^4 + 4x^3 - 12x^2$. Find the coordinates of the stationary points of f and determine their nature. For what values of x is the function increasing? For what values of k will $f(x) = 0$ have no solution?

(h) An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

(i) A rectangular box with a square base, an open top, and a volume of 216 in.^{3} is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

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